Choosing the Best: Decision- Making under Uncertainty

Allan J. Rossman
Dept of Statistics, Cal Poly – San Luis Obispo
arossman@calpoly.edu
https://askgoodquestions.blog
@allanjrossman

Outline

- Personal story
- Problem description
  - Predictions
- Enumeration analysis for small cases
  - Key insight
- Derivation, analysis of probability function
- Approximation analysis
- Asymptotic analysis
  - Remarkable result
- Extensions

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The problem

Your task is to hire a new employee

- You know how many candidates have applied (*n*)
- The candidates arrive in random order
- You interview candidates one at a time
- You can rank candidates after interviewing them
  - But you have no *a priori* sense of quality

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The problem

- After you have interviewed a candidate, you must decide immediately whether to hire
  - No follow-up interviews
  - No going back later
- Your task is to choose *the* best!
  - Any other result: you've failed

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Predictions

Predict the optimal probability of success when:

- *n* = 4?
- *n* = 12?
- *n* = 24?
- *n* = 500?
- *n* = 7,601,872,508?

Please record your predictions, and be ready to write lots of answers on your handout

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Enumeration analysis for small cases

- *n* = 1:

A: 1

Probability of success = 1!

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Enumeration analysis for small cases

- *n* = 2:

A: 12 B: 21

- Two strategies
  - Hire first candidate
  - Hire second candidate

Probability of success = .5

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Enumeration analysis for small cases

- *n* = 3:

A: 123 B: 132 C: 213
D: 231 E: 312 F: 321

- Seems like probability of success = 1/3
  - Can we do better?

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Enumeration analysis for small cases

- Key insight
  - We can *learn* from the first candidate
- Optimal strategy
  - Let one go by
  - Then hire first candidate who is the *best so far*

A: 123 B: 132 ***C: 213***
D: 231 ***E: 312*** F: 321

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Enumeration analysis for small cases

- n = 4:

A: 1234 B: 1243 C: 1324 D: 1342 E: 1423 F: 1432

G: 2134 H: 2143 I: 2314 J: 2341 K: 2413 L: 2431

M: 3124 N: 3142 O: 3214 P: 3241 Q: 3412 R: 3421

S: 4123 T: 4132 U: 4213 V: 4231 W: 4312 X: 4321

- Same form for optimal strategy
  - But should we let 1 go by or let 2 go by?

Enumeration analysis for small cases

Let 1 go by (**winners in bold**)

A: 1234 B: 1243 C: 1324 D: 1342 E: 1423 F: 1432

G: 2134 **H: 2143** **I: 2314** **J: 2341** **K: 2413** **L: 2431**

M: 3124 **N: 3142** O: 3214 P: 3241 **Q: 3412** R: 3421

S: 4123 **T: 4132** U: 4213 V: 4231 W: 4312 X: 4321

- Probability of success = 11/24 ≈ .4583

Enumeration analysis for small cases

Let 2 go by (<u>*winners underlined, in italics*</u>)

A: 1234 B: 1243 C: 1324 D: 1342 E: 1423 F: 1432

G: 2134 **H: 2143** <u>***I: 2314***</u> <u>***J: 2341***</u> <u>***K: 2413***</u> <u>***L: 2431***</u>

M: 3124 **N: 3142** <u>*O: 3214*</u> <u>*P: 3241*</u> <u>***Q: 3412***</u> R: 3421

S: 4123 **T: 4132** <u>*U: 4213*</u> <u>*V: 4231*</u> <u>*W: 4312*</u> X: 4321

- Probability of success = 10/24 ≈ .4167

Enumeration analysis for small cases

- n = 4:
- Optimal strategy
  - Let 1 go by, then hire first one who is best so far
  - Probability of success = 11/24 ≈ .4583
    - Going down, but not as quickly as most expect

Derivation of probability function

- Optimal strategy in general
  - Let a certain number go by
  - Then hire the first one you see who is best so far
- Let n be the number of candidates
- Let r be the position of the first "contender"
  - So r – 1 is the number that you let go by
- Let S denote the event that you succeed in choosing the best
- Let B_i denote the event that the best is in position i

Derivation of probability function

- For i = 1, …, r – 1: $\Pr(S \mid B_i) = 0$
- For i = r: $\Pr(S \mid B_r) = 1$
- For i = r + 1, …, n: $\Pr(S \mid B_i) = (r-1)/(i-1)$
  - Need best among those before overall best to be in the "training set"
    - In other words: Need best among 1, …, (i – 1) to occur within 1, …, (r – 1)
    - In other words: Want no "contenders" (before position i) to be better than all in "training set"

Derivation of probability function

$$\Pr(S)=\sum_{i=1}^{n}\Pr(S \mid B_i)\Pr(B_i)=\frac{1}{n}\sum_{i=1}^{n}\Pr(S \mid B_i)$$

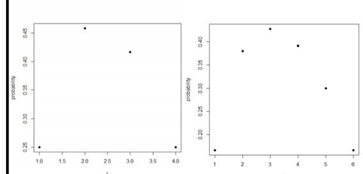
$$=\frac{1}{n}\left(1+\sum_{i=r+1}^{n}\frac{r-1}{i-1}\right)$$

$$=\begin{cases}\dfrac{1}{n} & r=1\\[2ex] \dfrac{r-1}{n}\displaystyle\sum_{i=r}^{n}\frac{1}{i-1} & r>1\end{cases}$$

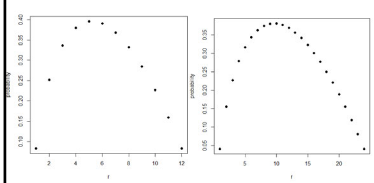
Analysis of probability function

- Run R program to calculate probabilities for
  - n = 4 (compare to our by-hand work)
  - n = 6
  - n = 12
  - n = 24
  - n = 50
  - n = 100
  - n = 500
  - n = 1000

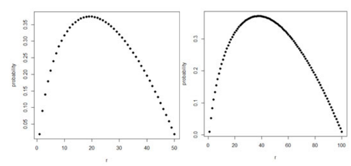
Analysis of probability function



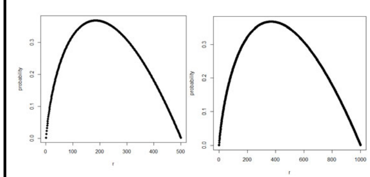
Analysis of probability function



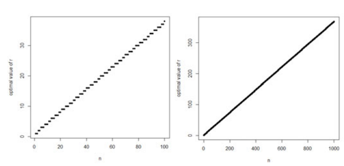
Analysis of probability function



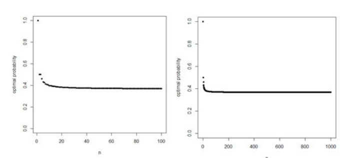
Analysis of probability function



Analysis of probability function



Analysis of probability function



Practice

	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y
1	4	8	7	4	7	4	2	9	12	1	5	3	12	1	5	3	1	12	3	6	3	6	8	5	7
2	12	1	5	3	12	1	5	3	11	11	10	7	8	5	11	11	10	7	1	12	4	8	7	4	3
3	1	12	3	6	3	6	8	5	7	4	2	9	6	10	12	1	9	2	4	8	10	7	1	12	10
4	5	3	9	2	9	2	4	8	1	12	3	6	3	6	8	5	2	9	6	10	2	9	6	10	4
5	6	10	12	1	1	12	3	6	8	5	11	11	5	3	9	2	12	1	5	3	12	1	5	3	11
6	7	4	2	9	6	10	12	1	5	3	9	2	9	2	4	8	6	10	12	1	1	12	3	6	8
7	11	11	10	7	8	5	11	11	6	10	12	1	1	12	3	6	4	8	7	4	7	4	2	9	12
8	3	6	8	5	11	11	10	7	2	9	6	10	2	9	6	10	3	6	8	5	11	11	10	7	1
9	8	5	11	11	5	3	9	2	9	2	4	8	10	7	1	12	7	4	2	9	6	10	12	1	5
10	10	7	1	12	4	8	7	4	3	6	8	5	11	11	10	7	8	5	11	11	5	3	9	2	9
11	2	9	6	10	2	9	6	10	4	8	7	4	7	4	2	9	11	11	10	7	8	5	11	11	6
12	9	2	4	8	10	7	1	12	10	7	1	12	4	8	7	4	5	3	9	2	9	2	4	8	2

Practice

	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y
1	4	8	7	4	7	4	2	9	12	1	5	3	12	1	5	3	1	12	3	6	3	6	8	5	7
2	12	1	5	3	12	1	5	3	11	11	10	7	8	5	11	11	10	7	1	12	4	8	7	4	3
3	1	12	3	6	3	6	8	5	7	4	2	9	6	10	12	1	9	2	4	8	10	7	1	12	10
4	5	3	9	2	9	2	4	8	1	12	3	6	3	6	8	5	2	9	6	10	2	9	6	10	4
5	6	10	12	**1**	**1**	12	3	6	8	5	11	11	5	3	9	2	12	**1**	5	**3**	12	**1**	5	**3**	11
6	7	4	**2**	9	6	10	12	**1**	5	3	9	**2**	9	2	**4**	8	6	10	12	1	**1**	12	3	6	8
7	11	11	10	7	8	5	11	11	6	10	12	1	**1**	12	3	6	4	8	7	4	7	4	2	9	12
8	3	6	8	5	11	11	10	7	2	9	6	10	2	9	6	10	3	6	8	5	11	11	10	7	**1**
9	8	5	11	11	5	3	9	2	9	2	4	8	10	7	1	12	7	4	2	9	6	10	12	1	5
10	10	7	1	12	4	8	7	4	3	6	8	5	11	11	10	7	8	5	11	11	5	3	9	2	9
11	2	9	6	10	2	9	6	10	4	8	7	4	7	4	2	9	11	11	10	7	8	5	11	11	6
12	**9**	**2**	4	8	10	**7**	**1**	12	**10**	**7**	**1**	12	4	**8**	7	**4**	**5**	3	**9**	2	9	2	**4**	8	2

Approximate analysis

- Recall:

$$\Pr(S) = \frac{r-1}{n}\sum_{i=r}^{n}\frac{1}{i-1}$$

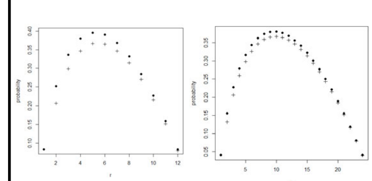
- For large values of *n*:

$$\sum_{i=r}^{n}\frac{1}{i-1} \approx \int_{r}^{n+1}\frac{dx}{x-1} = \ln\frac{n}{r-1}$$

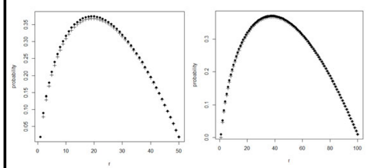
- And so:

$$\Pr(S) \approx \frac{r-1}{n} \times \ln\frac{n}{r-1}$$

Approximate analysis



Approximate analysis



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Asymptotic analysis

- Recall: $$\Pr(S) \approx \frac{r-1}{n} \times \ln \frac{n}{r-1} \equiv f(r)$$
- Goal: choose *r* to maximize *f*(*r*)
- Elementary calculus gives: $r^* = 1 + \frac{n}{e}$
- Optimal probability of success becomes:

$$\Pr(S) \approx \frac{r^* - 1}{n} \ln \frac{n}{r^* - 1} = \frac{1}{e} \approx .3679$$

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Remarkable result

- As *n* gets infinitely large
  - Optimal strategy: Let first 1/e (about 37%) go by, then hire first who is best so far
  - Optimal probability of success: 1/e, about 37%

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Other applications

- Hidden number game
- Finding a soul-mate

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References

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- Ferguson, T.S. (1989), "Who Solved the Secretary Problem?*", Statistical Science*, 4, 282-289.

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Thanks very much!

- Questions, comments?
  - arossman@calpoly.edu

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